Locating peaks of a Schrödinger equation with sign-changing nonlinearity

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\textbf{A B S T R A C T}

In this paper, we study the concentration phenomenon of a positive ground state solution of a nonlinear Schrödinger equation on $\mathbb{R}^N$. The coefficient of the nonlinearity of the equation changes sign. We prove that the solution has a maximum point at $x_0 \in \Omega^+ = \{x \in \mathbb{R}^N : Q(x) > 0\}$ where the energy attains its minimum.

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\textbf{1. Introduction}

In this paper, we are concerned with standing waves of the following nonlinear Schrödinger equation

\begin{equation}
\frac{i h}{\partial t} \psi = -\frac{h^2}{2m} \Delta_x \psi + V(x) \psi - Q(x) |\psi|^{p-1} \psi,
\end{equation}

where $m$ and $h$ are positive constants, $\psi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$, $V \in C(\mathbb{R}^N, \mathbb{R})$. The so-called standing waves are solutions of (1.1) of the form $\psi(x,t) = e^{-iEt}u(x)$, where $E$ is some real constant and $u(x)$ is real-valued. After suitably relabeling the parameters, $u(x)$ satisfies

\begin{equation}
-\varepsilon^2 \Delta u + V(x)u = Q(x) |u|^{p-1}u \quad \text{in } \mathbb{R}^N,
\end{equation}

where $\varepsilon > 0$ is a parameter.

If $V$ and $Q$ are positive, the problem has been studied in [1–4] and references therein. In particular, it is shown in [4] that positive ground state solutions of (1.2) concentrate at a global minimum point of the function $A(x) := V^{\frac{p-1}{2}}(x)Q^{-\frac{1}{2}}(x)$ as $\varepsilon \to 0$. Similar results are obtained in [1] for the case that both $V(x)$ and $Q(x)$ vanish at infinity. In [5], multiple solutions are found in relation to the set of critical points of $A(x)$. An important ingredient in studying the concentration phenomenon is the monotonicity of critical values with respect to the parameters related to the coefficients $V$ and $Q$.

On the other hand, if $Q$ is sign-changing, various existence results are obtained in [6,7] etc. The main difficulties are that firstly, the negative part $Q^-$ of $Q$ will push the energy level of the associated functional up, it then brings obstacles to verify the minimax geometry if the variational method is applied; secondly, the boundedness of Palais–Smale sequences is hard to show due to the presence of $Q^-$. Hence, special conditions are introduced to verify the condition, for instance, the “thickness” condition or a non-degeneracy condition on the set where $Q = 0$.

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In this paper, we assume in problem (1.2) that \( 1 < p < \frac{N+2}{N-2} \) and \( N \geq 3 \). The functions \( V \) and \( Q \) are bounded and continuous. \( V \) satisfies

\[
V(x) \geq V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0.
\]

Suppose the function \( Q \) changes sign, that is \( Q^+ \neq 0, Q^- \neq 0 \), where \( Q^+(x) = \max\{0, Q(x)\} \) and \( Q^-(x) = \min\{0, Q(x)\} \), and that

\[
\lim_{|x| \to \infty} Q(x) < 0,
\]

which implies that the set

\[
\Omega^+ := \{ x \in \mathbb{R}^N : Q(x) > 0 \}
\]

is bounded. We say a solution of (1.2) is a ground state solution if it has the least energy among all solutions. It is known that the problem

\[
-\Delta u + u = u^p \quad \text{in} \quad \mathbb{R}^N
\]

has a positive ground state solution. Such a solution is unique and radially symmetric. It also decays exponentially at infinity, see [8,9]. The positive ground state solution of (1.5) can actually be obtained as a minimizer of the problem

\[
c_\infty := \inf_{u \in \mathcal{N}_\infty} I_\infty(u),
\]

where

\[
I_\infty = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} u_{\infty}^{p+1} \, dx
\]

and

\[
\mathcal{N}_\infty = \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\}, \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx = \int_{\mathbb{R}^N} |u|^{p+1} \, dx \right\}.
\]

Define

\[
c(\xi) = \frac{V(\xi)^{\frac{p+1}{p+1-N}}}{Q(\xi)^\frac{p+1}{p+1-N}} c_\infty := \alpha(\xi) c_\infty
\]

for \( \xi \in \Omega^+ \). Our main result is stated as follows.

**Theorem 1.1.** For \( \varepsilon_n \to 0^+ \), there exist nonnegative ground state solutions \( \{u_n\} \) of (1.2) concentrating at the minimum point \( x_0 \in \Omega^+ \) of

\[
c(x_0) = \inf_{\xi \in \Omega^+} c(\xi),
\]

in the sense that \( u_n \) has a unique maximum point \( x_n \) satisfying \( x_n \to x_0 \) and for each \( \delta > 0 \) and \( n \) large

\[
\max_{|x-x_0| \leq \delta} u_n(x) \geq C,
\]

and

\[
u_n(x) \leq C \left| \frac{x - x_n}{\varepsilon_n} \right|^{1-N} \exp \left( \frac{x - x_n}{\varepsilon_n} \right)
\]

for \( x \in \mathbb{R}^N \), where \( C > 0 \) is a constant.

In the proof of Theorem 1.1, we will look for solutions of the problem

\[
-\Delta u + V(\varepsilon x)u = Q(\varepsilon x)|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N,
\]

by finding critical points of associated functional

\[
l_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(\varepsilon x)u^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} Q(\varepsilon x) u_{\varepsilon}^{p+1} \, dx
\]

defined on \( E := H^1(\mathbb{R}^N) \). By the hypotheses on \( V \) and \( Q \), for fixed \( \varepsilon > 0 \),

\[
\|u\| = \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + V(\varepsilon x)u^2) \, dx \right)^\frac{1}{2}
\]
is a norm equivalent to the usual one, and $I_\epsilon$ is a $C^1$ functional on $E$. There are three ways to describe the least energy of $I_\epsilon$ if $Q \geq 0$. One is given by the mountain pass theorem,

$$c_\epsilon = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} I_\epsilon(u),$$

(1.12)

where

$$\Gamma = \{ \gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e \}.$$

(1.13)

In particular, the second one is defined by

$$a_\epsilon = \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I_\epsilon(tu),$$

(1.14)

and the last one is the least energy value of solutions,

$$b_\epsilon = \inf_{u \in N_\epsilon} I_\epsilon(u),$$

(1.15)

where

$$N_\epsilon = \{ u \in E \setminus \{0\} : \langle I_\epsilon'(u), u \rangle = 0 \}.$$  

It was shown in [4] that if $Q^- \equiv 0$,

$$a_\epsilon = b_\epsilon = c_\epsilon.$$  

However, if $Q^- \neq 0$, it is not clear if (1.16) holds. We can recover just the inequality $b_\epsilon \leq c_\epsilon$ instead of equality in general because the strong effect of the negative part on the energy functional. We first show by the mountain pass theorem that $c_\epsilon$ is a critical value of $I_\epsilon$, then we show $b_\epsilon$ is achieved, that is, there exists a ground state solution of $I_\epsilon$. This will be done in Section 2. In Section 3, we obtain an upper bound for $c_\epsilon$. Finally, we investigate in Section 4 the limiting behavior of ground state solutions and prove Theorem 1.1.

2. Preliminaries and mountain pass

Let $v$ be a solution of (1.2), then $u(x) = v(\epsilon x)$ is a solution of

$$-\Delta u + V(\epsilon x)u = Q(\epsilon x)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N,$$

(2.1)

and vice versa. In this section, we investigate the existence of a nonnegative solution of problem (2.1).

**Lemma 2.1.** If $u_\epsilon$ is a nonnegative solution of (2.1) and

$$u_\epsilon(x_\epsilon) = \max_{x \in \mathbb{R}^N} u_\epsilon(x),$$

then $x_\epsilon \in \Omega^+$.  

**Proof.** Since $x_\epsilon$ is a maximum point of $u_\epsilon$, $\Delta u_\epsilon(x_\epsilon) \leq 0$. If $x_\epsilon \in \Omega^- = \{ x \in \mathbb{R}^N : Q(x) < 0 \}$, we would have

$$0 < -\Delta u_\epsilon + V(x_\epsilon)u_\epsilon = Q(x_\epsilon)|u_\epsilon(x_\epsilon)|^p \leq 0,$$

which is a contradiction. \hfill \Box

Using the mountain pass theorem [10], we look for critical points of the associated functional $I_\epsilon(u)$ of (2.1) in $E := H^1(\mathbb{R}^N)$.

**Proposition 2.2.** The functional $I_\epsilon$ has the mountain pass geometry. That is

(i) There exist $\rho > 0$ and $\alpha > 0$ such that

$$I_\epsilon|_{S_{\rho}} \geq \alpha.$$

(ii) There exists $e \in E$ such that $I_\epsilon(e) < 0$.

**Proof.** By the assumptions on $V$ and $Q$,

$$I_\epsilon(u) \geq \frac{1}{2} \|u\|^2 - C \|u\|^{p+1}.$$

Hence, (i) follows by choosing $\|u\|_{H^1} = \rho$ small.

Next, we show (ii). Let $\varphi \geq 0$ be a function with $\text{supp} \varphi \subset \Omega^+$. Then, $I_\epsilon(t\varphi) < 0$ for $t > 0$ large enough. We can fix $e = t_0\varphi$ such that $I_\epsilon(t_0\varphi) < 0$ for $t \geq 1$. The proof is complete. \hfill \Box

Now, we define $c_\epsilon$ as (1.12). By the mountain pass theorem, $c_\epsilon$ is a critical value of $I_\epsilon$ if $I_\epsilon$ satisfies (PS)$_{c_\epsilon}$ condition. By (PS)$_c$ condition we mean that any sequence $\{u_n\} \subset E$ satisfying $I_\epsilon(u_n) \to c$ and $I_\epsilon'(u_n) \to 0$, possesses a convergent subsequence.
**Proposition 2.3.** The functional $I_c$ satisfies (PS)$_c$ condition for $c \in \mathbb{R}$.

**Proof.** Let $\{u_n\}$ be a (PS)$_c$ sequence of $I_c$, namely,
\begin{equation}
I_c(u_n) \to c, \quad I'_c(u_n) \to 0.
\end{equation}
Then,
\[ c = \left( \frac{1}{2} - \frac{1}{p+1} \right) \|u_n\|^2 + o(1) \|u_n\|, \]
implicating that $\|u_n\|$ is uniformly bounded in $n$. We may assume that
\begin{align*}
&u_n \rightharpoonup u_0 \text{ in } H^1(\mathbb{R}^N), \\
&u_n \to u_0 \text{ in } L^{\frac{2N}{N-2}}(\mathbb{R}^N) \quad \text{for } 2 \leq \gamma < \frac{2N}{N-2},
\end{align*}
and
\[ u_n \to u_0 \text{ a.e. in } \mathbb{R}^N. \]
By Brézis–Lieb’s Lemma [11],
\[ o(1) + \langle I'_c(u_n), u_0 \rangle = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(\varepsilon x)u_n^2) \, dx - \int_{\mathbb{R}^N} Q(\varepsilon x)(u_n)^{p+1}_+ \, dx \\
= \langle I'_c(u_0), u_0 \rangle + \|u_n - u_0\|^2 - \int_{\mathbb{R}^N} Q^+(\varepsilon x)(u_n - u_0)^{p+1}_+ \, dx + \int_{\mathbb{R}^N} Q^- (\varepsilon x)(u_n - u_0)^{p+1}_+ \, dx. \]
Since $u_0$ is a solution of (2.1), $\langle I'_c(u_0), u_0 \rangle = 0$, we obtain
\begin{equation}
\|u_n - u_0\|^2 + \int_{\mathbb{R}^N} Q^+(\varepsilon x)(u_n - u_0)^{p+1}_+ \, dx = \int_{\mathbb{R}^N} Q^+(\varepsilon x)(u_n - u_0)^{p+1}_+ \, dx + o(1). \tag{2.3}
\end{equation}
For fixed $\varepsilon > 0$, $\Omega^+_\varepsilon$ is bounded, hence
\[ \int_{\mathbb{R}^N} Q^+(\varepsilon x)(u_n - u_0)^{p+1}_+ \, dx \to 0 \]
as $n \to +\infty$. This and (2.3) yield $u_n \to u_0$ in $H^1(\mathbb{R}^N)$. \qed

By the mountain pass theorem and Propositions 2.2 and 2.3, we see that there is $u_* \in E \setminus \{0\}$ satisfying
\begin{equation}
-\Delta u + V(\varepsilon x)u = Q(\varepsilon x)u_+^{p-1} \quad \text{in } \mathbb{R}^N. \tag{2.4}
\end{equation}
Multiplying both sides of (2.4) by $(u_*)_-$ and integrating by part, we find $u_* \geq 0$. Consequently, for each $\varepsilon > 0$, problem (2.1) has a nonnegative solution. Moreover, this implies that $\mathcal{N}_*^c$ is not empty. Now, for any minimizing sequence of $b_\varepsilon$, by Ekeland’s variational principle, we may assume that it is a (PS)$_b$ sequence for $I_c$. Proposition 2.3 implies that the sequence has a convergent subsequence, and thus $b_\varepsilon$ is achieved, the minimizer is a ground state solution of (2.1).

3. **Energy estimates**

In this section, we investigate the limiting behavior of the ground state solution $u_\varepsilon$ of (2.1) as $\varepsilon \to 0$. So we have
\[ I_c(u_\varepsilon) = b_\varepsilon. \]
For $\xi \in \Omega^+_\varepsilon$, we define on $E$ the functional
\[ l_\xi(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(\xi)u^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} Q(\xi)u_+^{p+1} \, dx, \]
which is $C^1$. Hence, we may define the associated Nehari manifold
\[ M_\xi := \{ u \in E \setminus \{0\} : (l'_\xi(u), u) = 0 \} \]
and the variational problem
\[ c(\xi) := \inf_{u \in M_\xi} l_\xi(u). \tag{3.1} \]
It is well known from [8] that \( c(\xi) \) is achieved provided that \( \xi \in \Omega^+ \) by a positive radial function \( u \), which satisfies
\[
-\Delta u + V(\xi)u = Q(\xi)u^p \quad \text{in } \mathbb{R}^N,
\]
and \( u \) is exponentially decaying at infinity. Let \( w(\lambda) = \frac{1}{\lambda} u(\frac{\lambda \cdot \cdot}{\mu}) \). Then, \( w \) satisfies
\[
-\Delta w + \frac{1}{\mu^2} V(\xi)w = \frac{1}{\lambda \mu^2} Q(\xi)\lambda^p w^p.
\]
Choosing \( \mu^2 = V(\xi), \lambda^{p-1} \frac{Q(\xi)}{V(\xi)} = 1 \), we see that \( w \) satisfies
\[
-\Delta w + w = w^p \quad \text{in } \mathbb{R}^N.
\]
The positive ground state solution of (3.3) is actually the minimizer of the problem
\[
c_\infty := \inf_{u \in A_\infty} I_\infty(u),
\]
where
\[
I_\infty = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1} \, dx
\]
and
\[
A_\infty = \left\{ u \in E \setminus \{0\}, \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx = \int_{\mathbb{R}^N} |u|^{p+1} \, dx \right\}.
\]
Therefore,
\[
c(\xi) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} Q(\xi) |u|^{p+1} \, dx = \left( \frac{1}{2} - \frac{1}{p+1} \right) \frac{V(\xi)^{p+1-\frac{2}{p}}}{Q(\xi)^{p+1-\frac{2}{p}}} \int_{\mathbb{R}^N} |w|^{p+1} \, dx,
\]
that is
\[
c(\xi) = \frac{V(\xi)^{p+1-\frac{2}{p}}}{Q(\xi)^{p+1-\frac{2}{p}}} c_\infty := \alpha(\xi) c_\infty
\]
for \( \xi \in \Omega^+ \). Let
\[
c_{\min} := \inf_{\xi \in \Omega^+} \alpha(\xi) c_\infty,
\]
and \( Q_M := \max_{x \in \mathbb{R}^N} Q(x) > 0 \). We define on \( E \) the functionals
\[
I_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)u^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} Q^+(\epsilon x)|u|^{p+1} \, dx,
\]
and
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_0 u^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} Q_M |u|^{p+1} \, dx.
\]
Then
\[
I_\epsilon(u) \geq J_\epsilon(u) \geq J(u).
\]
Let \( c \) be the mountain pass level of \( J \), that is
\[
c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} J(u)
\]
where \( \Gamma = \{ \gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e \} \). We define
\[
\mathcal{N} = \{ u \in E \setminus \{0\} : J'(u, u) = 0 \}.
\]
Since \( I_\epsilon \) and \( J \) have a critical point described by the mountain pass lemma, \( \mathcal{N} \) and \( \mathcal{N} \) are not empty. Let
\[
b = \inf_{u \in \mathcal{N}} J(u).
\]
Then, we have

**Proposition 3.1.** There holds

\[ b_c \geq b > 0. \]  \( (3.7) \)

Now, we estimate the upper bound of \( b_c \). It is sufficient to obtain an upper bound of \( c_\epsilon \). We note that \( \inf_{\xi \in \Omega^+} \alpha(\xi) \) is achieved. Let \( x_0 \in \partial \Omega^+ \) be the point such that \( \alpha(x_0) = \inf_{\xi \in \Omega^+} \alpha(\xi) \). Fix \( R > 0 \), we may assume \( B_{R+1}(x_0) \subset \Omega^+_R := \{ x \in \epsilon^{-1} \Omega^+ \} \) for \( \epsilon > 0 \). We will choose a particular \( \epsilon = \epsilon_R \) and construct a path \( \gamma(t) = te_R \) in \( \Gamma \) defined in (1.12) such that the following estimate holds for \( c_\epsilon = c_{\epsilon,R} \).

**Proposition 3.2.** There holds

\[ \limsup_{\epsilon \to 0} c_{\epsilon,R} \leq \alpha(x_0)c_\infty + o_\epsilon(1), \]  \( (3.8) \)

where \( o_\epsilon(1) \to 0 \) if \( R \to +\infty \).

**Proof.** It suffices to choose a particular \( \epsilon_R \) with \( \text{supp} \, \epsilon_R \subset \Omega^+_R \) and construct a path \( \gamma(t) = te_R \) in \( \Gamma \) defined in (1.12) such that

\[ \max_{t \geq 0} l_c(te_R) \leq \alpha(x_0)c_\infty + o_\epsilon(1) + o_\epsilon(1). \]

where \( o_\epsilon(1) \to 0 \) as \( \epsilon \to 0 \), which implies the result.

Now we select \( \epsilon_R \). Since \( Q\big|_{\Omega^+_0} = 0 \), \( x_0 \in \Omega^+ \). Fix \( R > 0 \), we assume \( B_{R+1}(x_0) \subset \Omega^+_R \) for \( \epsilon > 0 \) small. Let \( \eta_R \in C_\infty(\mathbb{R}^N) \), \( \eta_R \equiv 1 \) if \( x \in B_R(x_0) \) and \( \eta_R \equiv 0 \) in \( \Omega_{R+1}(x_0) \), \( 0 \leq \eta_R \leq 1, |\nabla \eta_R| \leq c_0 \). Let \( u \) be the unique positive ground state solution of (3.2) with \( \xi = x_0 \). Define \( v_R = \eta_R u \) and \( w_R(x) = u_R(x - \frac{x_0}{\epsilon}) \). Hence,

\[ l_c(t w_R) = \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla w_R|^2 + V(\epsilon x) w_R^2) \, dx - \frac{t^{p+1}}{p+1} \int_{\mathbb{R}^N} Q(\epsilon x)|w_R|^{p+1} \, dx. \]  \( (3.9) \)

Solving

\[ \frac{d l_c(t w_R)}{dt} = t \int_{\mathbb{R}^N} (|\nabla w_R|^2 + V(\epsilon x) w_R^2) \, dx - t^p \int_{\mathbb{R}^N} Q(\epsilon x)|w_R|^{p+1} \, dx = 0, \]

we obtain

\[ t_{c,R} := \left( \frac{\int_{\mathbb{R}^N} (|\nabla w_R|^2 + V(\epsilon x) w_R^2) \, dx}{\int_{\mathbb{R}^N} Q(\epsilon x)|w_R|^{p+1} \, dx} \right)^{\frac{1}{p+1}}. \]  \( (3.10) \)

Now

\[ \frac{d^2 l_c(t w_R)}{dt^2} \bigg|_{t=t_{c,R}} = \int_{\mathbb{R}^N} (|\nabla w_R|^2 + V(\epsilon x) w_R^2) \, dx - p t_{c,R}^{p-1} \int_{\mathbb{R}^N} Q(\epsilon x)|w_R|^{p+1} \, dx \]

\[ = (1 - p) \int_{\mathbb{R}^N} (|\nabla w_R|^2 + V(\epsilon x) w_R^2) \, dx < 0. \]  \( (3.11) \)

Therefore,

\[ \max_{t \geq 0} l_c(t w_R) = l_c(t_{c,R} w_R). \]  \( (3.12) \)

On the other hand,

\[ \int_{\mathbb{R}^N} (|\nabla w_R|^2 + V(\epsilon x) w_R^2) \, dx = \int_{\mathbb{R}^N} (|\nabla v_R|^2 + V(\epsilon x + x_0) v_R^2) \, dx \]

\[ = \int_{\mathbb{R}^N} (|\nabla v_R|^2 + V(x_0) v_R^2) \, dx + \int_{\mathbb{R}^N} (V(\epsilon x + x_0) - V(x_0)) v_R^2 \]

\[ =: l_1 + l_2. \]

We estimate

\[ |l_1 - \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x_0) u^2) \, dx| \leq C \int_{B_R(x_0)} (|\nabla u|^2 + u^2) \, dx + C \int_{B_{R+1}(x_0) \setminus B_R(x_0)} (|\nabla u|^2 + u^2) \, dx \]

\[ := o_\epsilon(1) \to 0 \]
as \( R \to \infty \). Now, fix \( R > 0 \), since \( V(\varepsilon x + x_0) \to V(x_0) \) for \( x \in B_R \) as \( \varepsilon \to 0 \), by the Lebesgue dominated convergence theorem, we deduce
\[
|I_2| \leq \int_{B_{R+1}(x_0)} |V(\varepsilon x + x_0) - V(x_0)| u^2 \, dx := o_\varepsilon(1) \to 0
\]
as \( \varepsilon \to 0 \). Therefore
\[
\int_{\mathbb{R}^N} \left( |\nabla w_R|^2 + V(\varepsilon x) w_R^2 \right) \, dx = \int_{\mathbb{R}^N} \left( |\nabla u|^2 + V(x_0) u^2 \right) \, dx + o_\varepsilon(1) + o_R(1).
\]
(3.13)
Similarly,
\[
\int_{\mathbb{R}^N} Q(\varepsilon x) |w_R|^{p+1} \, dx = \int_{\mathbb{R}^N} Q(x_0) |v_R|^{p+1} \, dx + \int_{\mathbb{R}^N} (Q(\varepsilon x + x_0) - Q(x_0)) |v_R|^{p+1} \, dx
= \int_{\mathbb{R}^N} Q(x_0) |u|^{p+1} \, dx + o_\varepsilon(1) + o_R(1).
\]
Consequently,
\[
t_{\varepsilon,R} = 1 + o_\varepsilon(1) + o_R(1).
\]
(3.14)
Now, we choose \( \varepsilon_R = t_{\varepsilon,R} w_R \), then \( \text{supp} \, \varepsilon_R \subset \Omega^+ \) for \( \varepsilon > 0 \) small and we deduce
\[
c_{\varepsilon,R} \leq \max_{t \geq 0} I_t(t \varepsilon_R) = I_{t_{\varepsilon,R} w_R} = \int_{\mathbb{R}^N} |\nabla u|^2 + V(x_0) u^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} Q(x_0) |u|^{p+1} \, dx + o_\varepsilon(1) + o_R(1)
= \alpha(x_0) c_{\infty} + o_\varepsilon(1) + o_R(1).
\]
(3.15)
It implies
\[
limsup_{\varepsilon \to 0} c_{\varepsilon,R} \leq \alpha(x_0) c_{\infty} + o_R(1).
\]
The proof is complete. \( \square \)

4. Concentration phenomenon

We know from Section 2 that for each \( \varepsilon > 0 \), there is a nonnegative solution \( u_\varepsilon \) of (2.1) satisfying
\[
I_{\varepsilon}(u_\varepsilon) = b_\varepsilon.
\]
By Propositions 3.1 and 3.2,
\[
0 < b_r \leq b_\varepsilon \leq c_{\varepsilon,R} \leq \inf_{\xi \in \Omega^+} c(\xi) + o(\varepsilon) + o_R(1),
\]
and
\[
limsup_{\varepsilon \to 0} b_\varepsilon \leq \limsup_{\varepsilon \to 0} c_{\varepsilon,R} \leq \inf_{\xi \in \Omega^+} c(\xi) + o_R(1).
\]
(4.1)
Notice that \( b_\varepsilon \) is independent of \( R > 0 \), we have
\[
limsup_{\varepsilon \to 0} b_\varepsilon \leq \inf_{\xi \in \Omega^+} c(\xi).
\]
(4.2)
Hence, there is a sequence \( \{\varepsilon_n\}, \varepsilon_n \to 0 \) such that
\[
0 < b \leq b_n := b_{\varepsilon_n} \to b_0 \leq \inf_{\xi \in \Omega^+} c(\xi) = \alpha(x_0) c_{\infty}
\]
as \( n \to 0 \). Corresponding to \( b_n \), we have a solution \( u_n \) of (2.1) such that \( b_n = I_{\varepsilon_n}(u_n) \). We now investigate the limiting behavior of \( \{u_n\} \). To this end, we consider the concentration function
\[
\bar{Q}(r) = \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} \rho_n(x) \, dx
\]
of
\[
\rho_n = |\nabla u_n|^2 + u_n^2.
\]
Using the concentration-compactness principle in [12], we will show that the sequence \( \{u_n\} \) is tight.
Proposition 4.1. Vanishing does not occur for \( \{u_n\} \).

Proof. If vanishing occurs, by the vanishing lemma, see [12] or [13],

\[
    u_n \to 0, \quad \text{as} \quad n \to \infty \quad \text{in} \quad L^\gamma (\mathbb{R}^N), \quad 2 < \gamma < \frac{2N}{N-2}.
\]

It would yield

\[
    0 = \left( \frac{1}{2} - \frac{1}{p+1} \right) \lim_{N \to \infty} Q_0 \int_{\mathbb{R}^N} |u_n|^{p+1} \, dx \\
    \geq \left( \frac{1}{2} - \frac{1}{p+1} \right) \lim_{N \to \infty} Q_0 (\varepsilon \rho x) |u_n|^{p+1} \, dx \\
    = \lim_{n \to \infty} c_n \geq c > 0
\]

as \( \varepsilon_n \to 0 \), which is a contradiction. \( \square \)

Proposition 4.2. Dichotomy does not occur for \( \{u_n\} \).

Proof. Suppose \( 0 < l = \lim_{n} \int_{\mathbb{R}^N} \rho_n \, dx \). If dichotomy occurs, there would exist \( 0 < \alpha < l \) such that, for \( \varepsilon > 0 \) and \( R \geq 1 \), there exist sequences \( \{y_n\} \subset \mathbb{R}^N \), \( \{R_n\} \subset \mathbb{R}^+ \) and \( \bar{R} > R \) satisfying \( R_0 < \bar{R} < \frac{l}{2} \), \( R_n < R_{n+1} \to \infty \) and \( Q(x) < 0 \) if \( |x| > R_0 \), there hold

\[
    \alpha - \varepsilon < \int_{|x-y_n| < \frac{R}{2}} \rho_n \, dx < \alpha + \varepsilon,
\]

and

\[
    \int_{|x-y_n| \leq 3R_n} \rho_n \, dx < \alpha + \varepsilon, \quad \int_{|x-y_n| \geq 3R_n} \rho_n \, dx > c - \alpha - \varepsilon.
\]

In particular,

\[
    \int_{\frac{R}{2} \leq |x-y_n| \leq 3R_n} \rho_n \, dx < 2\varepsilon. \tag{4.3}
\]

Let \( \psi \in C_0^\infty (\mathbb{R}^N) \) be such that \( \psi (x) = 1 \) if \( |x| \leq 1 \), \( \psi (x) = 0 \) if \( |x| \geq 2 \), \( \eta_n := 1 - \psi (x) \), and set

\[
    u_0^1 (x) = \psi \left( \frac{x-y_n}{R} \right) u_n =: \psi_n u_n (x), \quad u_0^2 (x) = \eta \left( \frac{x-y_n}{R} \right) u_n =: \eta_n u_n (x).
\]

Now, let us estimate \( \langle l' (u_n), u_n^2 \rangle \). First,

\[
    \left| \int_{R_0 \leq |x-y_n| \leq 2R_0} V (\varepsilon_n x) u_n^2 (u_n^2 - u_n) \, dx \right| \leq C \int_{\mathbb{R}^N} u_n^2 |u_n^2 - u_n| \, dx \\
    \leq C \int_{\mathbb{R}^N} \psi_n \eta_n |u_n|^2 \, dx \\
    < \varepsilon,
\]

that is,

\[
    \int_{\mathbb{R}^N} V (\varepsilon_n x) u_n u_n^2 \, dx = \int_{\mathbb{R}^N} V (\varepsilon_n x) (u_n^2)^2 \, dx + O (\varepsilon). \tag{4.4}
\]

Similarly,

\[
    \int_{\mathbb{R}^N} \nabla u_n \nabla u_n^2 \, dx = \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + O (\varepsilon) \tag{4.5}
\]

and

\[
    \left| \int_{\mathbb{R}^N} Q (\varepsilon_n x) (|u_n|^p u_n^2 - |u_n|^p u_n^p) \, dx \right| \leq c \int_{R_0 \leq |x-y_n| \leq 2R_0} |u_n|^p \, dx \\
    \leq c \int_{R_0 \leq |x-y_n| \leq 2R_0} |u_n|^2 + |u_n|^2 \, dx.
\]
Therefore,
\[
0 = \left( I_n^0, u_n^2 \right) = \int_{\mathbb{R}^N} (\nabla u_n \nabla u_n^2 + V(\varepsilon x) u_n u_n^2 - Q(\varepsilon x) |u_n^2|^{p+1}) \, dx
\]
\[
= \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(\varepsilon x)(u_n^2)^2 - Q(\varepsilon x) |u_n^2|^{p+1}) \, dx + O(\varepsilon).
\]
That is
\[
\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(\varepsilon x)(u_n^2)^2 - Q(\varepsilon x) |u_n^2|^{p+1}) \, dx = O(\varepsilon).
\]
(4.6)
Similarly,
\[
\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(\varepsilon x)(u_n^1)^2 - Q(\varepsilon x) |u_n^1|^{p+1}) \, dx = O(\varepsilon).
\]
(4.7)
Hence, we may find \( t_n^1, t_n^2 > 0 \) such that \( t_n^1 u_n^1, t_n^2 u_n^2 \in \mathcal{N}_{a_n} \) and
\[
t_n^1 = 1 + o(1), \quad t_n^2 = 1 + o(1).
\]
As a result,
\[
l_{a_n}(t_n^1 u_n^1) \geq b_{a_n}, \quad l_{a_n}(t_n^2 u_n^2) \geq b_{a_n}.
\]
It yields
\[
b_{a_n} = l_{a_n}(u_n) = l_{a_n}(t_n^1 u_n^1) + l_{a_n}(t_n^2 u_n^2) + O(\varepsilon) \geq 2b_{a_n} + O(\varepsilon)
\]
a contradiction provided \( n \) large since \( b_{a_n} \) is uniformly bounded below by a positive constant. \( \square \)

It follows that compactness necessarily takes place, i.e.,

**Proposition 4.3.** There exists \( \{y_n\} \subset \mathbb{R}^N \) such that for every \( \varepsilon > 0 \), there exists \( R > 0 \) such that
\[
\int_{B_R(y_n)} (|\nabla u_n|^2 + V(\varepsilon x)(u_n^2)^2) \, dx \geq l - \varepsilon.
\]
(4.8)
Let \( u_n(x) = u_n(x + y_n) \). Then, \( u_n \) is a solution of
\[
-\Delta u_n + V(\varepsilon x + \varepsilon y_n) u_n - Q(\varepsilon x + \varepsilon y_n) |u_n|^{p-1} u_n = 0.
\]
(4.9)

**Proposition 4.4.** The sequence \( \{\varepsilon_n y_n\} \) is bounded.

**Proof.** By (4.8), for every \( \varepsilon > 0 \) there exists \( R > 0 \) such that if \( n \) large
\[
\int_{\mathbb{R}^N \setminus B_R(0)} (|\nabla w_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n) w_n^2) \, dx = \int_{\mathbb{R}^N \setminus B_R(0)} (|\nabla u_n(x + y_n)|^2 + V(\varepsilon_n x + \varepsilon_n y_n) u_n(x + y_n)^2) \, dx
\]
\[
\leq C \int_{\mathbb{R}^N \setminus B_R(0)} (|\nabla u_n|^2 + (u_n)^2) \, dx < \varepsilon
\]
(4.10)
and
\[
\left| \int_{\mathbb{R}^N \setminus B_R(0)} Q(\varepsilon x + \varepsilon_n y_n) |u_n|^{p+1} \, dx \right| < \varepsilon.
\]
Hence,
\[
0 = \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n) w_n^2) \, dx - \int_{\mathbb{R}^N} Q(\varepsilon_n x + \varepsilon_n y_n) |u_n|^{p+1} \, dx
\]
\[
= \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n) w_n^2) \, dx - \int_{B_R(0)} Q(\varepsilon x + \varepsilon_n y_n) |u_n|^{p+1} \, dx + O(\varepsilon).
\]
(4.11)
If \(\varepsilon_n y_n \to \infty\), let \(y := \varepsilon_n x + \varepsilon_n y_n\) or \(x = \frac{y}{\varepsilon_n} - y_n\). For \(|x| \leq R\), we have \(|\varepsilon_n y_n| \leq R + |y|\). Hence, \(|y| \geq R_0\) if \(n\) large, which yields \(Q(\varepsilon_n x + \varepsilon_n y_n) < 0\). So we deduce from (4.11) that

\[
\int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n) w_n^2) \, dx = O(1),
\]

(4.12)

which contradicts the fact that

\[
\int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n) w_n^2) \, dx \to 1 > 0.
\]

Consequently, \(\{\varepsilon_n y_n\}\) is bounded. \(\Box\)

Finally, we prove the strong convergence of \(\{w_n\}\).

**Proposition 4.5.** We have that

\[
w_n \to w_0 \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{and} \quad c(x_0) = \inf_{\xi \in \Omega^+} c(\xi).
\]

(4.13)

**Proof.** Since \(\{w_n\}\) is bounded, we may assume that \(w_n \to w_0\) in \(H^1(\mathbb{R}^N)\), \(w_0 \geq 0\) and \(w_n \to w_0\) in \(L^p(\mathbb{R}^N)\) for any \(2 \leq p \leq 2^*\). By the tightness of \(w_n\) and the boundedness of \(\{\varepsilon_n y_n\}\) as well as (4.10), \(w_0 \neq 0\) and

\[
w_n \to w_0 \quad \text{in} \quad L^p(\mathbb{R}^N), \quad 2 \leq p \leq 2^*.
\]

(4.14)

Suppose \(y_n := \varepsilon_n y_n \to x^*\). By (4.14) and the elliptic regularity theory, \(w_n \to w_0\) in \(C^2_{loc}\) as \(n \to \infty\), and

\[
-\Delta w_0 + V(x^*) w_0 = Q(x^*) w_0^p, \quad \text{in} \quad \mathbb{R}^N.
\]

(4.15)

Hence, we deduce

\[
\inf_{\xi \in \Omega^+} c(\xi) \leq c(x^*) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_0|^2 + V(x^*) w_0^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} Q(x^*) w_0^{p+1} \, dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} (|\nabla w_0|^2 + V(x^*) w_0^2) \, dx
\]

\[
\leq \liminf_n \left[ \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(\varepsilon_n x + x_n) w_n^2) \, dx \right]
\]

\[
= \liminf_n \left[ \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(\varepsilon_n x) w_n^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} Q(\varepsilon_n x) w_n^{p+1} \, dx \right]
\]

\[
= \liminf_n b_{\varepsilon_n} \leq \inf_{\xi \in \Omega^+} c(\xi).
\]

So we conclude that

\[
c(x^*) = \inf_{\xi \in \Omega^+} c(\xi)
\]

(4.16)

and

\[
\lim_n \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(\varepsilon_n x + x_n) w_n^2) \, dx = \int_{\mathbb{R}^N} (|\nabla w_0|^2 + V(x^*) w_0^2) \, dx.
\]

(4.17)

By (4.10) and (4.14),

\[
\lim_n \int_{\mathbb{R}^N} V(\varepsilon_n x + x_n) w_n^2 \, dx = \int_{\mathbb{R}^N} V(x^*) w_0^2 \, dx.
\]

Hence,

\[
\lim_n \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx = \int_{\mathbb{R}^N} |\nabla w_0|^2 \, dx.
\]

This and (4.14) imply \(w_n \to w_0\) in \(E\). \(\Box\)

**Proof of Theorem 1.1.** The exponentially decaying law can be established as [3]. The assertion follows by Propositions 4.1–4.4. \(\Box\)
Acknowledgements

The authors would like to thank the referee for his/her valuable comments. The first author was supported by Fondecyt 7050253 and Fondecyt 1080439, and the second author was supported by grants N10961016 and 10631030 from the NNSF of China as well as 2009GZS0011 of NSF of Jiangxi.

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