Equivalent conditions for generalized contractions on (ordered) metric spaces

Jacek Jachymski
Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93-005 Łódź, Poland

A R T I C L E   I N F O
Article history:
Received 30 March 2010
Accepted 14 September 2010

MSC:
primary 47H09
47H10
54H25
secondary 26A15

Keywords:
Fixed point
Generalized contraction
Altering distance function
Metric space
Ordered metric space

A B S T R A C T
We establish a geometric lemma giving a list of equivalent conditions for some subsets of the plane. As its application, we get that various contractive conditions using the so-called altering distance functions coincide with classical ones. We consider several classes of mappings both on metric spaces and ordered metric spaces. In particular, we show that unexpectedly, some very recent fixed point theorems for generalized contractions on ordered metric spaces obtained by Harjani and Sadarangani [J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal. 72 (2010) 1188–1197], and Amini-Harandi and Emami [A. Amini-Harandi, H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, Nonlinear Anal. 72 (2010) 2238–2242] do follow from an earlier result of O’Regan and Petruşel [D. O’Regan and A. Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. 341 (2008) 1241–1252].

1. Introduction
Let $(X, d)$ be a complete metric space and $T$ be a selfmap of $X$. Following Rus [1] $T$ is called a Picard operator if $T$ has a unique fixed point $x_*$ and for any $x \in X$, $\lim_{n \to \infty} T^n x = x_*$. Given a function $\varphi$ from $\mathbb{R}_+$, the set of all nonnegative reals, into $\mathbb{R}_+$ such that $\varphi(t) < t$ for all $t > 0$, we say that $T$ is a $\varphi$-contraction if
\[
d(Tx, Ty) \leq \varphi(d(x, y)) \quad \text{for any } x, y \in X.
\]
(1)
The following generalization of the Banach fixed point theorem was obtained in 1968 by Browder [2]; if $\varphi$ is right continuous and nondecreasing, then any $\varphi$-contraction is a Picard operator. Subsequently, this result was extended in 1969 by Boyd and Wong [3], who observed that it sufficed to assume only the right-upper semicontinuity of $\varphi$. Independently, the following contractive condition was introduced by Krasnoselskiĭ et al. [4]:
\[
d(Tx, Ty) \leq d(x, y) - \eta(d(x, y)) \quad \text{for any } x, y \in X,
\]
(2)
where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\eta^{-1}(\{0\}) = \{0\}$. Clearly, this is a special form of the Boyd–Wong condition with $\varphi(t) := t - \eta(t)$ for $t \in \mathbb{R}_+$. Next, condition (2) was rediscovered in 2001 by Rhoades [5], who assumed additionally that $\eta$ is nondecreasing. (See also [6]; here $\eta$ is continuous and nondecreasing, and such that $\lim_{t \to \infty} \eta(t) = \infty$.)

On the other hand, in 1976 Delbosco [7] initiated a study of the following contractive condition with the so-called altering distance function $\psi$:
\[
\psi(d(Tx, Ty)) \leq \alpha \psi(d(x, y)) \quad \text{for any } x, y \in X \text{ and some } \alpha \in (0, 1),
\]
(3)
E-mail address: jachym@p.lodz.pl.