Minimal antiderivatives and monotonicity

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We consider settings in convex analysis which give rise to families of convex functions that contain their lower envelope. Given certain partial data regarding a subdifferential, we consider the family of all convex antiderivatives that comply with the given data. We prove that this family is not empty and, in particular, contains a minimal antiderivative under a fairly general assumption on the given data. It turns out that the representation of monotone operators by convex functions fits naturally in these settings. Duality properties of representing functions are also captured by these settings, and the gap between the Fitzpatrick function and the Fitzpatrick family is filled by this broader sense of minimality of the Fitzpatrick function.

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1. Introduction

Let $X$ be a Banach space, $X^*$ its dual, and let $\langle x^*, x \rangle$ denote the value of $x^* \in X^*$ at $x \in X$. We identify a set-valued operator $T : X \rightarrow 2^{X^*}$ with its graph $\{(x, x^*) \in X \times X^* \mid x^* \in T(x)\}$ and say that $T$ is proper if $T \neq \emptyset$. The domain of $T$ is defined to be the set $\text{dom} \ T = \{x \in X \mid T(x) \neq \emptyset\}$. The operator $T$ is monotone if

$$\langle x, x^* \rangle, \ (y, y^*) \in T \Rightarrow \langle y^* - x^*, y - x \rangle \geq 0.$$  

$T$ is said to be maximal monotone if it has no proper monotone extension (in $X \times X^*$). Close relationships between monotonicity and convexity were observed numerous times (see, for example, the summary in [1]). This has led to a search for even deeper, more precise relationships. The first author who associated a convex function with a monotone operator was Fitzpatrick in [2]. Given any operator $T : X \rightarrow 2^{X^*}$, this function is now called the Fitzpatrick function $F_T : X \times X^* \rightarrow (-\infty, +\infty]$. It is defined by

$$F_T(x, x^*) := \sup_{(s, s^*) \in T} \langle s^*, x \rangle + \langle s^*, s \rangle - \langle s^*, s \rangle, \quad (x, x^*) \in X \times X^*. \quad (1)$$

$F_T$ is clearly convex and lower semicontinuous. When $T$ is monotone it follows that $F_T(x, x^*) = \langle x^*, x \rangle$ for $(x, x^*) \in T$. When $T$ is maximal monotone it follows that $\langle \cdot, \cdot \rangle \leq F_T$ pointwise, with equality only at the points of $T$. Fitzpatrick also considered the family of functions that satisfy such conditions, now known as the Fitzpatrick family:

$$\mathcal{F}_T := \left\{ h : X \times X^* \rightarrow (-\infty, +\infty) \left| \begin{array}{c} h \text{ is convex and lower semicontinuous,} \\
\langle \cdot, \cdot \rangle \leq h \\
\langle \cdot, \cdot \rangle |_{T} = h|_{T} \end{array} \right. \right\}, \quad (2)$$

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