A counterexample to uniqueness of generalized characteristics in Hamilton–Jacobi theory

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ABSTRACT

The notion of generalized characteristics plays a pivotal role in the study of propagation of singularities for Hamilton–Jacobi equations. This note gives an example of nonuniqueness of forward generalized characteristics emanating from a given point.

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1. Introduction

We are in this note concerned with generalized characteristics for the Hamilton–Jacobi equation

\[ S_t + H(x, \nabla S) = 0 \quad \text{in } Q = (0, \infty) \times \mathbb{R}^n, \quad S(0, x) = S_0(x) \quad \text{in } \mathbb{R}^n, \]

in the multidimensional case \( n \geq 2 \). While the existence of generalized characteristics is well-known, the corresponding uniqueness problem is largely unsettled. The purpose of the present contribution is to manifest that forward generalized characteristics are nonunique, in general. The Hamiltonian \( H \) appearing in (1) is the Legendre–Fenchel transform of a Lagrangian \( L \). We assume the following conditions linking (1) to a problem in the calculus of variations.

(A) The Lagrangian \( L \) is from \( C^2(\mathbb{R}^n \times \mathbb{R}^n) \). It fulfills \( \nabla^2_L \ell(x, v) > 0 \) and \( L(x, v) \geq \ell(|v|) \) for all \( (x, v) \in \mathbb{R}^n \times \mathbb{R}^n \) where \( \ell(s)/s \to \infty \) as \( s \to \infty \). The Hamiltonian \( H \) is given by

\[ H(x, p) = \max_{v \in \mathbb{R}^n} (\langle p, v \rangle - L(x, v)), \quad (x, p) \in \mathbb{R}^n \times \mathbb{R}^n. \]

(B) The initial function \( S_0 \) is locally semiconcave, i.e., for each compact, convex set \( C \subset \mathbb{R}^n \) there exists \( \alpha > 0 \) such that \( S_0(x) - \alpha|x|^2/2 \) is a concave function of \( x \in C \). Moreover, \( S_0(x) \geq -K(1 + |x|) \) for some constant \( K \geq 0 \).

In generic terms, \( \nabla^2 f \) signifies the Hessian matrix of a function \( f \in C^2(\mathbb{R}^n) \). The notation \( \nabla^2 f > 0 \) means that \( \nabla^2 f(p) \) is a positive definite matrix for every \( p \in \mathbb{R}^n \). Condition (A) ensures that \( H \in C^2(\mathbb{R}^n \times \mathbb{R}^n) \) and \( \nabla^2_H > 0 \) in \( \mathbb{R}^n \times \mathbb{R}^n \). We consider the functional

\[ J^t(x) = S_0(x(0)) + \int_0^t L(\dot{x}(s), x(s)) ds, \quad x \in \mathcal{A}(t, x), \]

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